# SHAPE OPTIMIZATION PROBLEM FOR INCOMPRESSIBLE VISCOUS FLOW BASED ON OPTIMAL CONTROL THEORY

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## INTRODUCTION

The purpose of this study is to apply a formulation of shape optimization to a numerical simulation of the body located in an incompressible viscous flow. The state equation for this problem can be expressed by an incompressible Navier Stokes equation. The shape optimization is based on the optimal control theory. In an optimal control theory, a control value which makes phenomenon an optimal state can be obtained. In this theory, a performance function should be introduced. When the performance function is minimized, it is assumed that the state is optimized, and then the control value is obtained. In this study, the optimal state is defined by the fluid forces subjected to the body. The shape optimization can be formulated to find surface coordinates of the body to minimize the performance function. In case of optimal control problem with constraint condition, the performance function should be minimized satisfying the state equation. This problem can be transformed into the minimization problem without constraint condition by the Lagrangian multiplier or the adjoint equations using adjoint variables corresponding to state variables of the state equations. A mixed interpolation by a triangular bubble element is employed to the finite element approximation. The bubble function is stabilized by the artificial diffusion which is chosen equal to the stabilized parameter of the SUPG scheme by a stabilized control parameter. The Sakawa Shindo method which is an interative procedure saving the computation memory and has a slow convergency so that the surface coordinates of the body does not move suddenly to avoid the distortion of the finite element mesh is applied for minimizing the performance function. For a numerical example, a drag minimization problem of a body which initial shape is a circular cylinder is introduced.

## STATE EQUATION

Let $\Omega$ denote the spatial domain representing $x_i$ the coordinates associated with $\Omega$ at the time I$\in[t_0,t_f]$. Let $\Gamma$ denote the boundary of $\Omega$, supposing that an incompressible viscous fluid flow occupies $\Omega \times I$. The state equation of the flow can be written by the following incompressible Navier Stokes equation in the non dimensional form:

$$\dot{u}_i + u_j u_{i,j} + p_{,i} - \nu( u_{i,j} + u_{j,i} )_{,j} \quad 0 \quad \text{in} \ \ \Omega \times I, \tag{1}$$

$$u_{i,i} = 0 \quad \text{in} \ \ \Omega \times I \tag{2}$$

where, $u_i$ and $p$ are the velocity and pressure, $\nu$ is the inverse of Reynolds number ($\nu = 1/Re$), respectively.

Consider a typical problem described in **Fig.1**, in which a solid body $B$ with the boundary $\Gamma_B$ is laid in an external flow. Suppose that the boundary conditions, for this problem is given as:

$$u_i = (U, 0) \quad \text{on} \quad \Gamma_U \times I, \tag{3}$$

$$t_i \quad \{-p\delta_{ij} + \nu(u_{i,j} + u_{j,i})\}n_j = 0 \quad \text{on} \quad \Gamma_D \times I, \tag{4}$$

$$t_1 = 0, \ u_2 = 0 \quad \text{on} \quad \Gamma_S \times I, \tag{5}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_B \times I, \tag{6}$$

$$u_i(x_i, t_0) = u_i^0 \ (\text{with } u_{i,i}^0 = 0) \quad \text{in} \quad \Omega. \tag{7}$$

where $U$ is constant inflow velocity, $t_i$ is traction vector, $n_i$ is unit vector of outward normal for $\Gamma$, respectively.

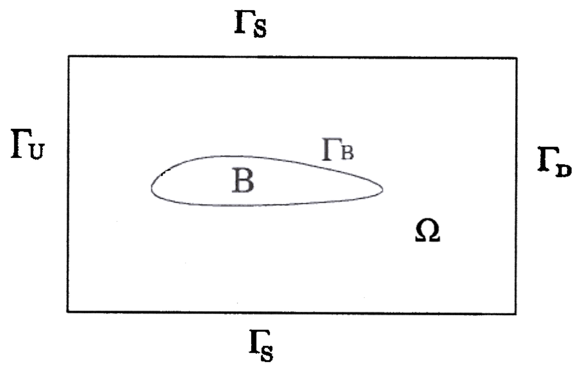


**Fig.1** Analytical domain and boundary condition

The fluid forces acting on the body $B$ are denoted by $F_i$. The fluid force $F_i$ are obtained by integrating the traction $t_i$, which is written as follows,

$$F_i = -\int_{\Gamma_B} t_i \, \mathrm{d}\Gamma. \tag{8}$$

## FORMULATION FOR SHAPE OPTIMIZATION

In case of optimal control problem with constraint conditions, the performance function should be minimized satisfying the state equations. This problem can be transformed into a minimization problem without constraint conditions by the Lagrangian multipliers or adjoint equations using adjoint variables to state variables of the state equations.

### Performance function

In this paper, a fluid force control problem is considered. The fluid force is directly used in the performance function. The performance function $J$ is defined by the square sum of the residual between fluid forces and objective values of fluid forces,

$$J = \frac{1}{2}\int_I \{q_1 (F_1 - \hat{F}_1)^2 + q_2 \ F_2 - \hat{F}_2)^2\} \mathrm{d}t, \tag{9}$$

where, $q_i$ is the weighting parameter, $\hat{F}_i$ is the objective value of fluid force which is a pre-asigned value. The state equations eq. (1) and (2) are the constraint conditions of the performance function $J$. The Lagrange multipliers for the eq. (1) and (2) are defined as the adjoint velocity $u_i^*$ and adjoint pressure $p^*$. The extended performance function $J^*$ can be obtained as follows:

$$
\begin{aligned}
J^* &= \frac{1}{2}\int_I \{q_1\,(F_1 - \hat{F}_1)^2 + q_2\,(F_2 - \hat{F}_2)^2\}\mathrm{d}t \\
&\quad - \int_I\int_\Omega u_i^*\{\dot{u}_i + u_j u_{i,j} + p_{,i} - \nu\,(u_{i,j} + u_{j,i})_{,j}\}\mathrm{d}\Omega\,\mathrm{d}t + \int_I\int_\Omega p^* u_{i,i}\mathrm{d}\Omega\,\mathrm{d}t.
\end{aligned}
$$

**Adjoint Equation**

To minimize the performance function $J$, the gradient of the performance function $J$ with respect to geometrical surface coordinates $x_i^c$ should be introduced. The optimal control problem with the constraint condition of eq. (1) and (2) results in solving a stationary condition of the extended performance function $J^*$ instead of the original performance function $J$. The necessary condition for the stationary condition is that the first variation of the extended performance function $J^*$ vanishes.

$$\delta J^* = 0.$$

Taking an integration by parts for each necessary term, the first variation of the performance function $\delta J$ yields as follows:

$$
\begin{aligned}
\delta J^* &= \delta u_i \frac{\partial J^*}{\partial u_i} + \delta p \frac{\partial J^*}{\partial p} + \delta u_i^* \frac{\partial J^*}{\partial u_i^*} + \delta p^* \frac{\partial J^*}{\partial p^*} + \delta x_i \frac{\partial J^*}{\partial x_i} \\
&= -\int_I\int_\Omega \delta u_i\{-\dot{u}_i^* + u_{j,i}\,u_j^* - u_j\,u_{i,j}^* + p^*,i - \nu(\,u_{i,j}^* + u_{j,i}^*)_{,j}\}\mathrm{d}\Omega\,\mathrm{d}t + \int_I\int_\Omega \delta p\, u_{i,i}^*\mathrm{d}\Omega\mathrm{d}t \\
&\quad + \int_I\int_{\Gamma_U} \delta t_i\, u_i^*\mathrm{d}\Gamma\,\mathrm{d}t + \int_I\int_{\Gamma_S} \delta t_2\, u_2^*\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_I\int_{\Gamma_B} \delta t_1\,\{u_1^* - q_1(F_1 - \hat{F}_1)\}\mathrm{d}\Gamma\,\mathrm{d}t + \int_I\int_{\Gamma_B} \delta t_2\,\{u_2^* - q_2(F_2 - \hat{F}_2)\}\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad - \int_I\int_{\Gamma_D} \delta u_i\, s_i\mathrm{d}\Gamma\,\mathrm{d}t - \int_I\int_{\Gamma_S} \delta u_1\, s_1\mathrm{d}\Gamma\,\mathrm{d}t - \int_I\int_{\Gamma_B} \delta x_i\, s_j\, u_{j,i}\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad - \int_\Omega \delta u_i\,(x_i,\, t_f)\, u_i^*(x_i,\, t_f)\mathrm{d}\Omega \\
&\quad 0,
\end{aligned}
$$

where $s_i$ is,

$$s_i = \{u_j u_i^* - p^*\,\delta_{ij} + \nu(u_{i,j}^* + u_{j,i}^*)\}n_j$$

Following relation is used,

$$\delta u_i = u_{i,j}\delta x_j\,.$$

Setting each term equal to zero to satisfy the optimal condition, following equations can be obtained,

$$-\dot{u}_i^* + u_{j,i}\,u_j^* - u_j\,u_{i,j}^* + p_{,i}^* - \nu(\,u_{i,j}^* + u_{j,i}^*)_{,j} = 0 \quad \text{in } \Omega \times I,$$

$$u^*_{i,i} = 0 \quad \text{in} \quad \Omega \times I, \tag{16}$$
$$u^*_i = 0 \quad \text{on} \quad \Gamma_U \times I, \tag{17}$$
$$s_i = 0 \quad \text{on} \quad \Gamma_D \times I, \tag{18}$$
$$s_1 = 0,\ u^*_2 = 0 \quad \text{on} \quad \Gamma_S \times I, \tag{19}$$
$$u^*_i = (q_1\,(F_1 - \hat{F}_1),\, q_2\,(F_2 - \hat{F}_2)) \quad \text{on} \quad \Gamma_B \times I, \tag{20}$$
$$u^*_i(t_f) = 0 \quad \text{in} \quad \Omega. \tag{21}$$

When the state and adjoint equation are solved, the gradient of the performance function related to the surface coordinates of the body can be obtained as follows,

$$\frac{\partial J}{\partial x_i} = -\int_I \int_{\Gamma_B} s_j\, u_{j,i} \mathrm{d}\Gamma\, \mathrm{d}t.$$

## MINIMIZATION

The Sakawa–Shindo method is applied for the minimization. In this method a modified performance function $K$ which a penalty term is added to the performance function is introduced.

$$K = J^{*(\iota)} + \frac{1}{2}\int_I \int_{\Gamma_B} W^{(\iota)}\{x_i^{(\iota+1)} - x_i^{(\iota)}\}^2 \mathrm{d}\Gamma\, \mathrm{d}t,$$

where $\iota$ is the iteration number for minimization, $x_i$ is the surface coordinates of the body. Let $x_i$ be the optimal solution, the derivative of the $K$ respect to $x_i$ is vanished. The renewed surface coordinates of the body is calculated by

$$W^{(\iota)}x_i^{(\iota+1)} - W^{(\iota)}x_i^{(\iota)} - \frac{\int_I \int_{\Gamma_B} s_j\, u_{j,i} \mathrm{d}\Gamma\, \mathrm{d}t}{\int_I \mathrm{d}t}$$

The following algorithm can be introduced.

1. Select initial surface coordinates $x_i^{(0)}$ in $\Omega$
2. Solve $u_i^{(0)}$ , $p^{(0)}$ by eqs. (1) , (2) in $\Omega$ from $t_0$ to $t_f$
3. Solve $u_i^{*(0)}$ , $p^{*(0)}$ by eqs. (15) , (16) in $\Omega$ from $t_f$ to $t_0$
4. Compute $x_i^{(\iota)}$ by eq. (24)
5. Solve $u_i^{(\iota)}$ , $p^{(\iota)}$ by eqs. (1) , (2) in $\Omega$ from $t_0$ to $t_f$
6. Compute $J^{(\iota)}$
7. If { $(J^{(\iota)})$ - $(J^{(\iota-1)})$} $< 0$ then $W^{(\iota)} = 2.0W^{(\iota)}$ go to 4.
   else $W^{(\iota)} = 0.9W^{(\iota)}$ go to 8.
8. IF $| x_i^{(\iota)} - x_i^{(\iota-1)} | < \epsilon$ then stop
   else solve $u_i^{*(\iota)}, p^{*(\iota)}$ by eqs. (15) (16) in $\Omega$ from $t_f$ to $t_0$
   go to 4.

### CONCLUSION

A numerical example of the shape optimization of a circular cylinder located in an incompressible viscous flow is analyzed. **Fig.2** and **Fig.3** show the analytical domain and finite element mesh, respectively. The total number of nodes and elements are 1636 and 3116. In this numerical cxamplc, thc diffusion dominant flow which Rcynolds numbcr was 10.0 is analyzcd. **Fig.4** shows the initial and the computed optimal shape of the objective body.

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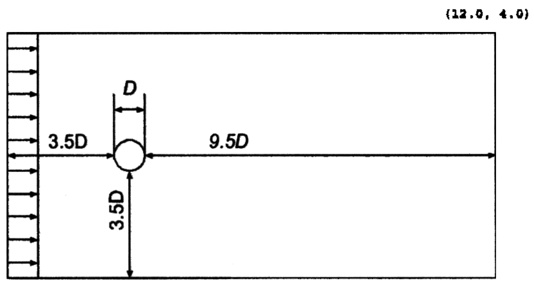


**Fig.2** Analytical domain

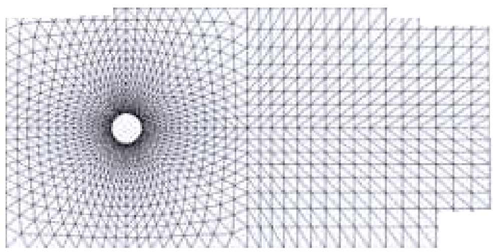

**Fig.3** Finite element mesh

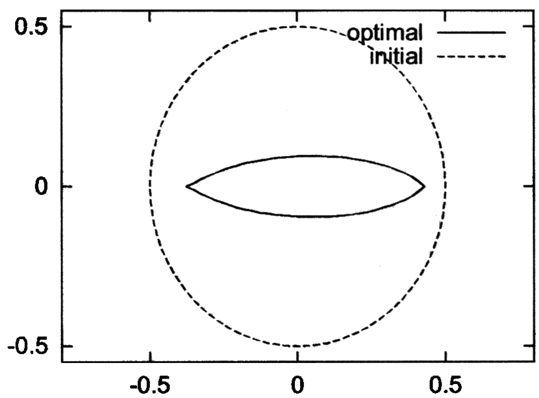


**Fig.4** Optimal and initial shape