Filling a Gap in Multiple Scattering Theory

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Abstract

Horizontal incidence and reflection by a plane-parallel atmosphere is investigated. A peculiar discontinuity of the reflected intensity is discussed. Several interesting properties of the bidirectional reflection function are presented, together with some applications.

1 Introduction

One of the best studied problems in multiple scattering theory is the reflection of radiation by a plane–parallel medium filled with independently scattering particles and illuminated at the top by a parallel beam of radiation, see e.g. [1]-[6]. In most publications the limiting case in which both the direction of incidence and that of reflection become grazing is not treated at all, or only touched upon. To fill this gap in multiple scattering theory we embarked some time ago in an investigation of this intriguing case. This has resulted in two papers, [7] and [8], dealing with the intensity (radiance) and polarization, respectively, of the reflected radiation. Here we focus on properties of the bidirectional reflection function.

2 Theory for horizontal directions

We consider a plane–parallel atmosphere or similar medium filled with randomly oriented particles that scatter radiation independently and without change of wavelength. The medium may be vertically inhomogeneous and semi–infinite or finite with or without a reflecting surface underneath. Suppose a parallel beam of radiation with net flux, πF_0, per unit area normal to itself is incident on every point of the top of the medium. Ignoring polarization we write the (specific) intensity of the reflected radiation emerging at the top of the medium in the form

$$I^{\mathrm{t}}(\mu, \mu_0, \phi - \phi_0) = \mu_0 R(\mu, \mu_0, \phi - \phi_0) F_0, \quad (1)$$

with $\arccos \mu$ the angle of the direction of the reflected light with the upward normal, $\arccos \mu_0$ the angle of the direction of the incoming radiation with the downward normal, ϕ and ϕ_0 the corresponding azimuth angles and $R(\mu, \mu_0, \phi - \phi_0)$ the bidirectional reflection function (BRF). It should be noted that $0 \leq \mu \leq 1$ and $0 \leq \mu_0 \leq 1$, where $\mu = 0$ means grazing reflection and $\mu_0 = 0$ grazing incidence.

Before presenting properties of the BRF we summarize some of the main results of [7], in which rigorous proofs are given. For the medium under consideration we have

$$\lim_{\mu_0\to 0} I_n^{\mathrm{t}}(\mu,\mu_0,\phi-\phi_0) = 0 \qquad (\mu \geq 0 \text{ and } n \geq 2), \tag{2}$$

$$\lim_{\mu,\mu_0\to 0} I^{\mathrm{t}}(\mu,\mu_0,\phi-\phi_0) = \lim_{\mu,\mu_0\to 0} I_1^{\mathrm{t}}(\mu,\mu_0,\phi-\phi_0), \tag{3}$$

$$\lim_{\mu,\mu_0\to 0} I^{\mathrm{t}}(\mu,\mu_0,\phi-\phi_0) = \frac{a^{\mathrm{t}}}{4(c+1)} Z^{\mathrm{t}}(\cos(\phi-\phi_0))F_0, \tag{4}$$

where n denotes the order of scattering, a^{t} the albedo of single scattering at the top, $Z^{\mathrm{t}}(\cos\Theta)$ the phase function at the top with Θ the scattering angle, and c a number that depends on how μ and μ_0 approach zero. More precisely, if we approach the origin of a Cartesian co–ordinate system, with μ_0 as the abscissa and μ as the ordinate, along a curve given by the function $\mu = g(\mu_0)$, the slope at the origin of $g(\mu_0)$ is c (see Fig. 1). Thus, for a straight line we have $\mu = c\mu_0$. For all curves we have $0 \leq c \leq \infty$, since μ and μ_0 are nonnegative. Therefore, for grazing incidence and reflection only first order scattering at the top of the medium contributes to the reflected intensity, but it has a peculiar discontinuity. The factor $1/(c+1)$ may take any value in the closed interval $[0,1]$, depending on how μ and μ_0 tend to zero. It is zero if we first let μ_0 tend to zero and then do the same with μ, but it is unity if μ is the first to become zero.

An important consequence of Eq. 4 is that, for a given value of c, the way in which the reflected intensity depends on azimuth in grazing incidence and reflection is proportional to the way in which the phase function at the top depends on the scattering angle. A similar statement was made by Minnaert in 1935 [9], but he did not mention the occurrence of a discontinuity, nor did he provide a rigorous proof.

We can now use the results for the intensity of the reflected light in the case of grazing incidence and reflection to derive properties of the BRF. Combining Eqs. 1 and 4 we find

$$\lim_{\mu,\mu_0\to 0} \mu_0 R(\mu,\mu_0,\phi-\phi_0) = \frac{a^{\mathrm{t}}}{4(c+1)} Z^{\mathrm{t}}(\cos(\phi-\phi_0)). \tag{5}$$

Multiplying both sides of Eq. 1 by μ/μ_0 leads to

$$\lim_{\mu,\mu_0\to 0} \mu R(\mu,\mu_0,\phi-\phi_0) = \frac{a^{\mathrm{t}}c}{4(c+1)} Z^{\mathrm{t}}(\cos(\phi-\phi_0)). \tag{6}$$

Consequently, a peculiar discontinuity occurs, when the BRF is multiplied by μ or μ_0. If $c = 1$ the two limits in Eqs. 5 and 6 are the same. Generally, the one is obtained from the other by replacing c by $1/c$. Since both limits are bounded it is clear that $\mu\mu_0 R(\mu,\mu_0,\phi-\phi_0) = 0$ if $\mu = \mu_0 = 0$.

By adding Eqs. 5 and 6 we obtain the simple relation

$$\lim_{\mu,\mu_0\to 0} (\mu+\mu_0) R(\mu,\mu_0,\phi-\phi_0) = \frac{a^{\mathrm{t}}}{4} Z^{\mathrm{t}}(\cos(\phi-\phi_0)), \tag{7}$$

which is independent of how μ and μ_0 tend to zero. Hence, adding the two peculiar discontinuities results in no discontinuity. We have thus obtained a rigorous proof of Eq. 7 for the general case of an inhomogeneous atmosphere which is semi–infinite or bounded by a reflecting surface. Eq. 7 has been reported without a rigorous proof for a semi–infinite homogeneous atmosphere by e.g. [3].

For natural and realistic model particles the right–hand side of Eq. 7 is always positive. Therefore, we must have

$$\lim_{\mu,\mu_0\to 0} R(\mu,\mu_0,\phi-\phi_0) = \infty, \tag{8}$$

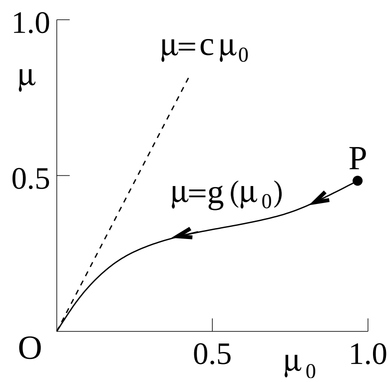

Figure 1: A point, P, approaches the origin, O, along a curve given by $\mu = g(\mu_0)$, which has a slope, c, at O with respect to the positive μ_0-axis. The tangent of the curve at O has also been drawn.

irrespective of the way in which this limit is taken, since any other result would be in conflict with Eq. 7. So the BRF itself has a discontinuity for $\mu = \mu_0$, but not a peculiar one. This is illustrated in Fig. 2 for a homogeneous, non–absorbing, semi–infinite atmosphere with isotropic scattering and $\mu = \mu_0$. As shown by [1] and later by [2] we have in this very simple case

$$R(\mu, \mu, \phi - \phi_0) = \frac{1}{8\mu} H^2(\mu), \tag{9}$$

where $H(\mu)$ is a well–known function that can be calculated by solving an integral equation and increases from 1 for $\mu = 0$ to 2.9 for $\mu = 1$. Figure 2 is based on a table (for $\varphi(\mu) = H(\mu)/2$) given by [10]. Apparently the factor $1/\mu$ is the main cause of the strong increase of the BRF in Fig. 2 as μ tends to zero. For comparison Chandrasekhar's function $S(\mu, \mu_0, \phi - \phi_0)$ [2] for $\mu = \mu_0$ is also shown in Fig. 2, illustrating its simpler behavior for small values of μ as compared to the BRF.

It should not be assumed that the discontinuity of the BRF for $\mu = \mu_0 = 0$ will never cause any problems upon integration. For example, in the case of isotropic scattering in a homogeneous, semi-infinite medium we readily find from the definition of the H-function

$$\int_0^1 R(\mu, \mu_0, \phi - \phi_0) d\mu = \frac{1}{2\mu_0} \left[H(\mu_0) - 1 \right], \tag{10}$$

which tends to infinity if μ_0 tends to zero [4].

3 Applications

The results presented in Sect. 2 can be used for various applications, as shown by the following examples.
1. Checking formulae valid for general μ and μ_0 in multiple scattering theory by letting both tend to zero.
2. Similarly, for numerical results, even for complicated models of atmospheres and oceans.
3. Interpolation of the BRF for small values of μ and μ_0, e.g. by first multiplying the BRF by $(\mu + \mu_0)$.
4. Using Eq. 7, approximate values for the phase function in the upper part of a cloud deck or aerosol layer can be obtained from observations for near grazing incidence and reflection. When this is done for a sufficient number of azimuthal angles integration over these angles yields an approximate value for the albedo of single scattering in the top layers, since the spatial average of the phase function equals unity. The necessary observations can be done, for instance, by a detector at a mountain top or in an airplane.

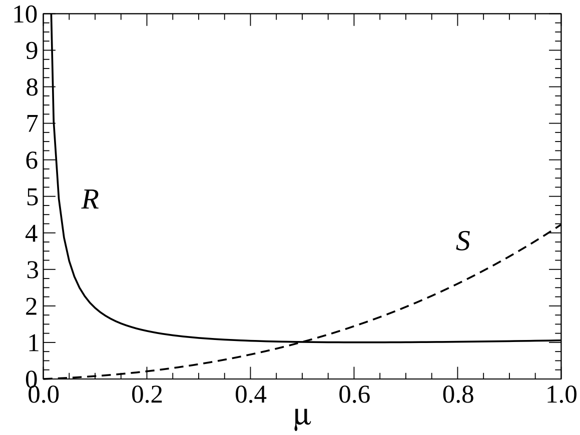

Figure 2: Bidirectional reflection function (R) and Chandrasekhar's function (S) of a non–absorbing, homogeneous, semi–infinite atmosphere with isotropic scattering in case $\mu = \mu_0$. Here, $S = 4\mu^2 R$.

5. Numerous approximation formulae for the BRF of plane-parallel media have been proposed. It is clear now that some of these cannot be accurate for near grazing incidence and reflection. This holds especially for the well-known "Lambert reflection law", which implies that the BRF would remain constant instead of tend to infinity when the directions of incidence and reflection become more and more horizontal.

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