Matrix model of inhomogeneous medium with generalized birefringence

Sergey N.Savenkov,[1] Konstantin E.Yushtin,[1] Ranjan S. Muttiah,[2] Viktor V. Yakubchak[1]

[1] *Taras Shevchenko Kiev National University, Department of Radiophysics, 64, Volodymyrska street, 01033 Kyiv, Ukraine, e-mail: sns@univ.kiev.ua*
[2] *Texas Christian University, Department of Geology, Fort Worth, Texas*

Abstract

We derive the Mueller matrix for single scattering from inhomogeneous medium characterized by simultaneous linear and circular birefringence. Simulations for I_{min}/I_{max} showed a simple dependence on wavelength of incident radiation.

1 Introduction

The Mueller matrix is a rich source of information about the properties of media investigated in a wide variety of disciplines [1-3]. And unique experimental methods are being rapidly developed to measure the matrix. The challenge is to interpret the measured Mueller matrix, and relate the matrix elements to observable phenomena. A polarization model for anisotropic media aids in these interpretations.

Mueller matrices for homogeneous anisotropic media are well known [4]. There are four basic anisotropies characterizing the homogeneous anisotropy of deterministic media: linear and circular dichroism, and linear and circular birefringence. Inhomogeneous media can depolarize the incident radiation, and, therefore, can not be directly described in terms of these four basic anisotropy properties. In [5], we derived and analyzed the Mueller matrix model for the rough plate parallel slab with linear birefringence. The main goal this paper is to derive the single scatter Mueller matrix model for an inhomogeneous medium with simultaneous linear and circular birefringence.

2 Theory

The geometry of the problem is given in Fig. 1. The object under discussion is a slab of anisotropic medium located in the $z=0$ plane. Inhomogeneity of the slab is specified by variation of its thickness $h(\boldsymbol{\rho})$ which sets the conditions for single scattering; $h(\boldsymbol{\rho})$ is assumed to follow a known statistical model – a uniform Gaussian process:

$$f(h)=\left(2\pi\sigma_h^2\right)^{-1/2}\exp\left\{-\left(h-\bar{h}\right)^2/2\sigma_h^2\right\} \tag{1}$$

with mean thickness $\bar{h}$, mean-square deviation σ_h, and correlation coefficient between screen thickness at two points given by:

$$\gamma_h(\boldsymbol{\rho}_-)=\frac{\overline{h(\boldsymbol{\rho}_1)\cdot h(\boldsymbol{\rho}_2)}}{\sigma_h^2}=\exp\left\{-\frac{\boldsymbol{\rho}_-^2}{\rho_0^2}\right\}, \tag{2}$$

where, the distance between points is given by: $\boldsymbol{\rho}_- = \boldsymbol{\rho}_2-\boldsymbol{\rho}_1$. The correlation radius is ρ_0.

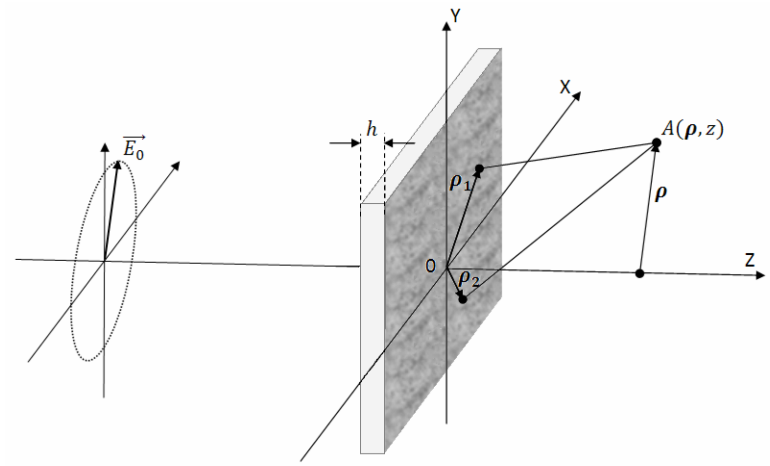

Fig. 1. Geometry for anistropic medium.

Anisotropy of the medium is given by its polarization eigen states, and the corresponding eigen values. Interaction of radiation with such medium is described by its Jones matrix written in eigen coordinate system (UOV) by:

$$\mathbf{J}^{eigen} = \begin{bmatrix} \exp(-i\phi_u) & 0 \\ 0 & \exp(-i\phi_v) \end{bmatrix} = \begin{bmatrix} g_u & 0 \\ 0 & g_v \end{bmatrix} \tag{3}$$

Where, $g_{u,v}$ are the corresponding eigen values which are the complex transmittance coefficients of radiation in eigen polarization states.

We will consider the case when fast eigen polarization of the studied object is associated with basis vector OU. We assume that field distribution of incident radiation to be Gaussian in the plane normal to its propagation direction, with the center of the beam located in the $z = 0$ plane:

$$\mathbf{E}^{in}(\boldsymbol{\rho}) = \mathbf{E}^{in} \exp\{-\boldsymbol{\rho}^2 / a^2\}, \tag{4}$$

Here, $\mathbf{E}^{in}$ denotes Jones vector in the center of the beam; a is the beam's radius. It has been shown in [5], that the Mueller matrix in the eigen coordinate system in the far field limit is (refer to [5] for definition of variables):

$$\mathbf{M}^{eigen} = \begin{pmatrix} \Phi_{11}+\Phi_{22} & \Phi_{11}-\Phi_{22} & 0 & 0 \\ \Phi_{11}-\Phi_{22} & \Phi_{11}+\Phi_{22} & 0 & 0 \\ 0 & 0 & \Phi_{12}+\Phi_{21} & i(\Phi_{12}-\Phi_{21}) \\ 0 & 0 & -i(\Phi_{12}-\Phi_{21}) & \Phi_{12}+\Phi_{21} \end{pmatrix}, \tag{5}$$

where:

$$\Phi_{uv}(\boldsymbol{\rho}, z) = \frac{1}{2}\left(\frac{kaw}{2z}\right)^2 \Phi_{uv}^{brf}\left[(1-\eta_{uv})\exp\left\{-\left(\frac{k\boldsymbol{\rho}w}{2z}\right)^2\right\} + \frac{\eta_{uv}}{\tilde{\sigma}_{uv}^2 w^2 + 1}\exp\left\{-\frac{1}{\tilde{\sigma}_{uv}^2 w^2 + 1}\left(\frac{k\boldsymbol{\rho}w}{2z}\right)^2\right\}\right];$$

$$\Phi_{uv}^{\mathrm{brf}} = \exp\left\{ik(n_u - n_v)\bar{h} - k^2\sigma_h^2(n_u - n_v)^2/2\right\};\ w = \left\{\frac{1}{2a^2} + \frac{a^2k^2}{8z^2}\right\}^{-1/2}$$

$$\sigma_{uv}^2 = k^2\sigma_h^2(n_u - 1)(n_v - 1),\ \tilde{\sigma}_{uv}^2 = \frac{\sigma_{uv}^2}{\eta_{ij}\rho_0^2},\ \eta_{uv} = 1 - \exp\left(-\sigma_{uv}^2\right)$$

In the general case, the Jones matrix in the laboratory coordinate system can be presented as:

$$\mathbf{J}^{\mathrm{lab}} = g_1\mathbf{B}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{B}^{-1} + g_2\mathbf{B}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{B}^{-1} = \sum_{u=1}^{2} g_u\boldsymbol{\beta}^u\ , \tag{6}$$

where, $\boldsymbol{\beta}^u = \mathbf{B}\begin{pmatrix}\delta_{u,1} & 0\\ 0 & \delta_{u,2}\end{pmatrix}\mathbf{B}^{-1}$.

If eigenvectors χ_1 and χ_2, ($\chi = E_2/E_1$) are known, then the transformation matrix $\mathbf{B}$ is:

$$\mathbf{B} = \frac{1}{\sqrt{\chi_2 - \chi_1}}\begin{pmatrix}1 & 1\\ \chi_1 & \chi_2\end{pmatrix}. \tag{7}$$

The Jones (as well as Mueller-Jones) matrix model of homogeneous anisotropic medium characterized by simultaneous linear and circular birefringence is defined by the first Jones' equivalence theorem [6,7]:

$$\mathbf{J}_{Lin}(\alpha,\delta)\,\mathbf{J}_{Cir}(\varphi) = \begin{bmatrix}\cos^2\alpha + \sin^2\alpha\exp(-i\delta) & \cos\alpha\sin\alpha(1 - \exp(-i\delta))\\ \cos\alpha\sin\alpha(1 - \exp(-i\delta)) & \cos^2\alpha\exp(-i\delta) + \sin^2\alpha\end{bmatrix}\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\phi & \cos\varphi\end{bmatrix}. \tag{8}$$

Then, the medium's eigenvectors $\begin{bmatrix}1 & \chi_{1,2}\end{bmatrix}^T$ can be calculated as:

$$\chi_{1,2} = \frac{-\xi \pm \eta}{\kappa}, \tag{9}$$

where,

$$\begin{gathered}\xi = \sin\frac{\delta}{2}\cos(2\alpha - \phi),\quad \eta = \sqrt{1 - \left(\cos\frac{\delta}{2}\cos\phi\right)^2}\\ \kappa = \sin\frac{\delta}{2}\sin(2\alpha - \phi) - i\cos\frac{\delta}{2}\sin\phi\end{gathered} \tag{10}$$

and matrices $\boldsymbol{\beta}^u$ from Eq.(6) are:

$$\boldsymbol{\beta}^1 = \frac{1}{2}\begin{pmatrix}1 + \dfrac{\xi}{\eta} & \dfrac{\kappa}{\eta}\\ \dfrac{\eta^2 - \xi^2}{\eta\kappa} & 1 - \dfrac{\xi}{\eta}\end{pmatrix};\quad \boldsymbol{\beta}^2 = \frac{1}{2}\begin{pmatrix}1 - \dfrac{\xi}{\eta} & -\dfrac{\kappa}{\eta}\\ \dfrac{\xi^2 - \eta^2}{\eta\kappa} & 1 + \dfrac{\xi}{\eta}\end{pmatrix}. \tag{11}$$

And finally, from Eqs.(5) and (6) after some algebra, the Mueller matrix model, in the laboratory reference, for the inhomogeneous medium with generalized birefringence in single scattering case for large inhomogeneities given by $\sigma_h > \sqrt{7}/2\pi\left\lfloor (n_1 - n_2)^{-1}\lambda\right\rfloor$ [5], is:

$$\mathbf{M} = \begin{pmatrix}\Phi_{11} + \Phi_{22} & \dfrac{\xi}{\eta}(\Phi_{11} - \Phi_{22}) & \dfrac{\mathrm{Re}\,\kappa}{\eta}(\Phi_{11} - \Phi_{22}) & \dfrac{\mathrm{Im}\,\kappa}{\eta}(\Phi_{11} - \Phi_{22})\\ \dfrac{\xi}{\eta}(\Phi_{11} - \Phi_{22}) & \dfrac{\xi^2}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{\xi\,\mathrm{Re}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{\xi\,\mathrm{Im}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22})\\ \dfrac{\mathrm{Re}\,\kappa}{\eta}(\Phi_{11} - \Phi_{22}) & \dfrac{\xi\,\mathrm{Re}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{(\mathrm{Re}\,\kappa)^2}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{\mathrm{Re}\,\kappa\,\mathrm{Im}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22})\\ \dfrac{\mathrm{Im}\,\kappa}{\eta}(\Phi_{11} - \Phi_{22}) & \dfrac{\xi\,\mathrm{Im}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{\mathrm{Re}\,\kappa\,\mathrm{Im}\,\kappa}{\eta^2}(\Phi_{11} + \Phi_{22}) & \dfrac{(\mathrm{Im}\,\kappa)^2}{\eta^2}(\Phi_{11} + \Phi_{22})\end{pmatrix}. \tag{12}$$

3 Discussions

As can be seen from Eq.(12), the matrix $\mathbf{M}$ is singular i.e., $\mathbf{M}$ exhibits dependence of output intensity on input polarization. From the fact that matrix $\mathbf{M}$ is non-deterministic, there are no input polarizations for which the intensity of output radiation is equal to zero. Maximum and minimum values of output intensities are obtained for input radiation with polarizations describing by the following Stokes vectors respectively:

$$\mathbf{S}^{\max,\min} = \left[\xi^2 + |\kappa|^2 \quad \pm\xi \quad \pm\mathrm{Re}(\kappa) \quad \pm\mathrm{Im}(\kappa)\right]^T . \tag{13}$$

The ratio of minimum and maximum output intensities is therefore:

$$\frac{I_{\min}}{I_{\max}} = \frac{\Phi_{11}}{\Phi_{22}} \tag{14}$$

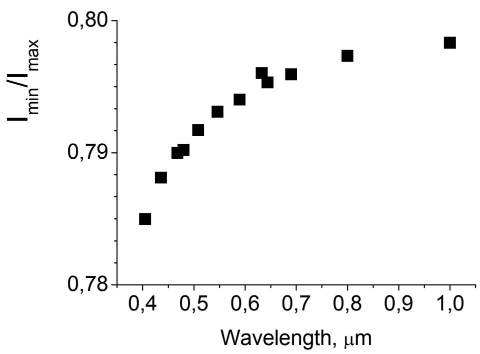

Figure 1. Dependence of ratio $I_{\min}/I_{\max}$ versus wavelength for TeO_2

Figure 1 presents the results of simulations performed to study the dependence of ratio $I_{\min}/I_{\max}$ versus wavelength from 0.4 to $1.0\ \mu m$. The data for refractive indices for this simulation was taken from [8] for paratellurite TeO_2.

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