Pattern equation method based on Wilcox representation

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Abstract

The variant of the pattern equation method, based on representation of a wave field by Atkinson-Wilcox expansion, is proposed for the solution of wave diffraction problem on compact scatterers. The basic integral-operator equation of the method is deduced and it's algebraization is obtained. Limitations on scatterer geometry, at which algebraic system can be resolved by a reduction method, are established. The analytical solution of plane wave diffraction problem on a sphere is obtained and it is shown, that it merges into the known classical solution at some special value of one of the method parameters. The suggested approach is illustrated by numerical example.

1 Introduction

The pattern equation method (PEM), for the first time suggested by the author in 1992 [1,2], subsequently has been successfully applied to the solution of the wide range of the waves scattering and propagation problems. At present moment about 30 papers dedicated to PEM application to the solution of various diffraction theory problems have been published in the leading journals. Some aspects of the method were also applied by other authors (see, for example, [3-5]).

Original variant of PEM is based on the plane wave representation of diffracted field by Sommerfeld-Weil integral [1, 2, 6]. The method is universal enough, and the algorithms based on it converge very fast.

However, the original method is poorly suitable for the solution of diffraction problems when the characteristic sizes of a scatterer are much greater than wavelength. The approach allowing to overcome this difficulty is developed below. The new approach is based on the wave field representation by a Atkinson-Wilcox series [7]. It is shown [8], that this series, converge in the area $r > 2\sigma/k$, where r is radial spherical coordinate, σ is a growth parameter of the scattering pattern [8,9] and k is a wavenumber. However, when the scatterer sizes are much greater than wavelength, Atkinson-Wilcox series, which are reduced to the first few summands, can be treated as asymptotic expansion [10].

2 The statement of the problem and its solution

So, lets consider a scalar diffraction problem on a compact scatterer, bounded by a surface S. We assume for definiteness, that Dirichlet boundary condition is satisfied on S. Thus, we seek a wave field function $u^1(\vec{r})$, as the solution of the following problem:

$$\Delta u^1 + k^2 u^1 = 0, \quad \vec{r} \in \mathbf{R}^3 \setminus \bar{D}, \quad (u^0 + u^1)\Big|_S = 0, \tag{1}$$

where D is an area inside S, u^0 is an incident (primary) wave. Function u^1 should also satisfy a Sommerfeld condition of radiation on infinity [11].

In spherical coordinates (r,θ,φ) function $u^1(\vec{r})$ has the following representation [7]:

$$u^1(r,\theta,\varphi) = \frac{e^{-ikr}}{kr}\sum_{j=0}^{\infty}\frac{g_j(\theta,\varphi)}{(kr)^j}, \tag{2}$$

where $g_0(\theta,\varphi) \equiv g(\theta,\varphi)$ is the pattern of wave field (the scattering pattern), and functions $g_j(\theta,\varphi)$ for $j > 0$ are determined by recurrent relations

$$g_j(\theta,\varphi)=\frac{i}{2j}[j(j-1)g_j(\theta,\varphi)+\mathrm{D}g_j(\theta,\varphi)],\quad j=1,2,3,..., \tag{3}$$

in which $\mathrm{D}\equiv\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ is the Beltrami operator [11,12].

We can combine the relations Eq. (2) and Eq. (3) using operator W (we call it Wilcox operator) so, that $u^1(r,\theta,\varphi)=W\left[g(\theta,\varphi)\right]$.

In case of Dirichlet boundary condition the following representation is valid

$$g(\alpha,\beta)=-\frac{k}{4\pi}\int_S\frac{\partial u}{\partial n}\bigg|_S\exp\left(ikr\big|_S\cos\gamma\right)ds, \tag{4}$$

here $u=u^0+u^1$ is a full field, $\frac{\partial}{\partial n}$- differentiation in the direction of external normal to S, $\cos\gamma=\sin\alpha\sin\theta\cos(\beta-\varphi)+\cos\alpha\cos\theta$, $r\big|_S=\rho(\theta,\varphi)$ is the equation of a surface S in spherical coordinates.

Let's introduce the following notations:

$$v(\theta,\varphi)=-\frac{k}{4\pi}d_nu,\quad v^0(\theta,\varphi)=-\frac{k}{4\pi}d_nu^0,\quad v^1(\theta,\varphi)=-\frac{k}{4\pi}d_nu^1, \tag{5}$$

where $d_nu\equiv\kappa\rho\frac{\partial u}{\partial n}\Big|_S=\left(\rho^2\sin\theta\frac{\partial u}{\partial r}-\rho'_\theta\sin\theta\frac{\partial u}{\partial\theta}-\frac{\rho'_\varphi}{\sin\theta}\frac{\partial u}{\partial\varphi}\right)\Bigg|_{r=\rho(\theta,\varphi)}$, $\kappa=\sqrt{\left(\rho^2+\rho_\theta'^2\right)\sin^2\theta+\rho_\varphi'^2}$.

Using these notations, representation (4) can be rewritten as follows

$$g(\alpha,\beta)=\int_0^{2\pi}\int_0^{\pi}v(\theta,\varphi)\exp\left(ik\rho(\theta,\varphi)\cos\gamma\right)d\theta d\varphi. \tag{6}$$

Combining Eq. (6) and Wilcox operator introduced above, finally we have

$$g(\alpha,\beta)=g^0(\alpha,\beta)-\frac{k}{4\pi}\int_0^{2\pi}\int_0^{\pi}d_nW\left[g(\theta,\varphi)\right]\exp\left(ik\rho(\theta,\varphi)\cos\gamma\right)d\theta d\varphi. \tag{7}$$

Here $g^0(\theta,\varphi)$ is the integral similar to Eq. (6), where the function $v^0(\theta,\varphi)$ is used instead of $v(\theta,\varphi)$.

Thus, the relation Eq. (7) is the sought integral-operator equation of PEM.

Let's algebraize the problem. For this purpose we shall expand function $g(\theta,\varphi)$ on some basis. For example, using the spherical harmonics, we shall have $g(\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{nm}P_n^m(\cos\theta)e^{im\varphi}$, $g_j(\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{nm}^jP_n^m(\cos\theta)e^{im\varphi}$. Using Eq. (3) it is easy to establish, that $a_{nm}^j=\frac{\left(n+\frac{1}{2},j\right)}{(2i)^j}a_{nm}$, where $\left(n+\frac{1}{2},j\right)=\begin{cases}\frac{1}{j!}\prod_{s=1}^{j}\left[n(n+1)-s(s-1)\right], j=1,2,...\\ 1,\quad j=0\end{cases}$ is Hankel symbol [13]. Using the notation

$$\frac{e^{-ikr}}{kr}\sum_{j=0}^{J}\frac{\left(n+\frac{1}{2},j\right)}{(2ikr)^j}\equiv h_n^J(kr), \tag{8}$$

expansion Eq. (2) can be rewritten as follows

$$u^{1J}(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{nm}h_n^J(kr)P_n^m(\cos\theta)e^{im\varphi}. \tag{9}$$

The index "J" means, that in relation Eq. (9) only first (J+1) summands of the sum Eq. (8) are kept, therefore it is now approximate. From Eq. (9) and Eq. (5) we have

$$d_nu^{1J}(\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{nm}\left[k\rho^2(\theta,\varphi)h_n^{J\prime}(k\rho)P_n^m(\cos\theta)\sin\theta-\rho'_\theta h_n^J(k\rho)\frac{dP_n^m}{d\theta}\sin\theta-\frac{im}{\sin\theta}\rho'_\varphi h_n^J(k\rho)P_n^m(\cos\theta)\right]e^{im\varphi}. \tag{10}$$

3 Bodies of revolution

Let's consider, for example, a case of a body of revolution, i.e. a situation, when $\rho(\theta,\varphi)=\rho(\theta)$. Let also u^0 be the field of a plane wave, i.e.

$$u^0 \equiv e^{-ikr\cos\gamma_0} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n(2n+1)\frac{(n-m)!}{(n+m)!}j_n(kr)P_n^m(\cos\theta)P_n^m(\cos\theta_0)e^{im(\varphi-\varphi_0)},$$

where $\cos\gamma_0=\sin\theta_0\sin\theta\cos(\varphi_0-\varphi)+\cos\theta_0\cos\theta$, θ_0,φ_0 are incident angles. In this case the equation Eq. (7) is reduced to the following algebraic system

$$a_{nm}^J = a_{nm}^0 + \sum_{\nu=|m|}^{\infty} G_{nm,\nu m}^J a_{\nu m}^J, \quad n=0,1,2,...;\, m=0,\pm1,...\pm n, \tag{11}$$

where $G_{nm,\nu m}^J = -i^n\dfrac{(2n+1)}{2}\dfrac{(n-m)!}{(n+m)!}\delta_{\mu m}\displaystyle\int_0^{\pi}[k^2\rho^2(\theta)h_\nu^{J\prime}(k\rho)P_\nu^m(\cos\theta)\sin\theta -$

$$-k\rho'_\theta h_\nu^J(k\rho)\frac{dP_\nu^m}{d\theta}\sin\theta]j_n(k\rho)P_n^m(\cos\theta)d\theta, \tag{12}$$

$$a_{nm}^0 = i^n\frac{(2n+1)}{2}\frac{(n-m)!}{(n+m)!}e^{-im\varphi_0}\int_0^{\pi}j_n(k\rho)P_n^m(\cos\theta)\sum_{\nu=|m|}^{\infty}(-i)^\nu(2\nu+1)\frac{(\nu-m)!}{(\nu+m)!}P_\nu^m(\cos\theta_0)\times$$

$$\times[k^2\rho^2(\theta)j_\nu{}'(k\rho)P_\nu^m(\cos\theta)\sin\theta - k\rho'_\theta j_\nu(k\rho)\frac{dP_\nu^m}{d\theta}\sin\theta]d\theta. \tag{13}$$

For sphere ($\rho(\theta)=a$) at $\theta_0=\varphi_0=0$ from Eq. (12) and Eq. (13) we obtain

$$G_{nm,\nu\mu}^J = -\delta_{\mu m}\delta_{\nu n}i^n k^2a^2 j_n(ka)h_n^{J\prime}(ka),\; a_{nm}^0 = -k^2a^2(2n+1)j_n(ka)j'_n(ka)\delta_{m0}. \tag{14}$$

Solving system Eq. (11) using Eq. (14), we obtain

$$a_{nm}^J = -\frac{k^2a^2(2n+1)j_n(ka)j'_n(ka)\delta_{m0}}{1+i^nk^2a^2j_n(ka)h_n^{J\prime}(ka)}. \tag{15}$$

The obtained solution (taking into account the remark after the formula Eq. (9)) is approximate. The solution Eq. (15), as well as initial relations Eqs. (9) - (14), become exact, if in (8) J is replaced by n. Thus expression $i^{n+1}h_n^J(kr)$ turns into spherical Hankel function $h_n^{(2)}(kr)$, and an expression Eq. (15) becomes

$$a_{nm} = -i(2n+1)\frac{j_n(ka)}{h_n^{(2)}(ka)}\delta_{m0}, \tag{16}$$

which coincides with classical result [14].

Figure 1 shows the scattering patterns of sphere with radius *ka*=21, calculated using the exact solution Eq. (16) (a continuous curve at *N*=40, a dotted curve at *N*=20) and approximate solution Eq. (15) (a dashed curve). *N* is the maximal number of value *n* in series for the scattering pattern. Parameter *J* in Eq. (15) has been set to 20. Good agreement between exact and approximate (at *J*=20) results is observed.

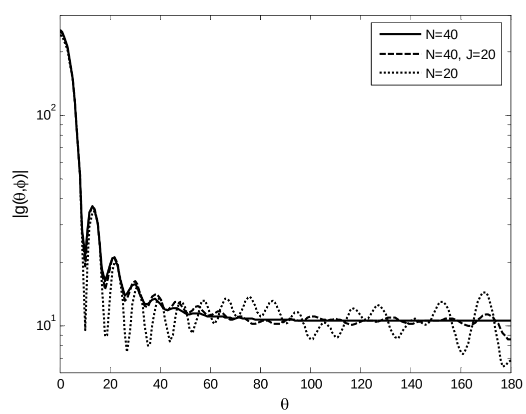

Figure 1: The scattering patterns of sphere with radius ka=21.

4 Conclusion

Thus, we obtained an already known variant of PEM, if in relations Eqs. (9) - (12) we replace J by n. However, fixing J on any finite value we achieve an alternative approach, allowing, as we just demonstrated, to solve (although approximately) problems of waves scattering by bodies with sizes considerably larger than incident field wavelength. The proposed approach can be easily extended to other kinds of boundary conditions, and also on vector scattering problems.

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