STABILITY OF FREE CONVECTION BETWEEN TWO CO-AXIAL CYLINDERS FILLED WITH A POROUS MEDIUM

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The theme of convection in enclosures filled with saturated porous media is a very popular one in the literature. The reason is the abundant applications, such as heat transfer in geothermal energy systems, storage of radioactive nuclear waste materials, pollutant dispersion in aquifers[1-2].

Various situations were examined in relation with the convection heat transfer in a porous medium between two vertical concentric cylinders[3-5]. For example, a parametric study has been performed[4] to evaluate mixed convection heat transfer in a porous medium between two vertical concentric cylinders for a constant temperature outer and a insulated inner boundary conditions. The authors of that paper claim on the fact that their simulation is related to the feasibility to extract energy from the liquid magma region located near the earth's surface. Bau and Torrance[6] analyzed convection in the annulus between vertical coaxial cylinders, and they found the preferred mode of convection is asymmetric. Charrier-Mojtabi et al[7] presented a numerical and experimental study of two-dimensional convection.

A linear stability analysis is performed in the present paper for the free convection between two co-axial cylinders filled with a porous medium.

BASIC EQUATIONS

We take a cylindrical polar co-ordinates system $(\bar{r},\theta,\bar{z})$ with the $\bar{z}$–axis along the vertical axis of the cylinders and put $R_1 = R$ and $R_2 = R + 2L$. In usual notations, the basic equations are:

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{u})+\frac{1}{\bar{r}}\frac{\partial \bar{v}}{\partial \theta}+\frac{\partial \bar{w}}{\partial \bar{z}}=0 \qquad (1)$$

$$\bar{u}=-\frac{K}{\mu}\frac{\partial \bar{p}}{\partial \bar{r}},\ \bar{v}=-\frac{K}{\mu}\frac{1}{\bar{r}}\frac{\partial \bar{p}}{\partial \theta},\ \bar{w}=\frac{K}{\mu}\left(\frac{\partial \bar{p}}{\partial \bar{z}}-\rho g\right) \qquad (2a\text{-}c)$$

$$\sigma\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{T}}{\partial \bar{r}}+\frac{\bar{v}}{\bar{r}}\frac{\partial \bar{T}}{\partial \theta}+\bar{w}\frac{\partial \bar{T}}{\partial \bar{z}}=\alpha_m\bar{\nabla}^2\bar{T} \qquad (3)$$

where $\bar{\nabla}^2=\frac{\partial^2}{\partial \bar{z}^2}+\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial}{\partial \bar{r}}\right)+\frac{1}{\bar{r}^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacean and the Boussinesq hypothesis was used: $\rho=\rho_0\left[1-\beta\left(\bar{T}-\bar{T}_m\right)\right]$, with $\bar{T}_m$ a mean temperature defined below.. We eliminate pressure $\bar{p}$ from Eqs. (2)-(4) to have:

$$\frac{\partial \bar{w}}{\partial \bar{r}}-\frac{\partial \bar{u}}{\partial \bar{z}}=\frac{gK\beta}{\nu}\frac{\partial \bar{T}}{\partial \bar{r}} \qquad (4)$$

$$\frac{\partial \bar{v}}{\partial \bar{r}}-\frac{\bar{v}}{\bar{r}}-\frac{1}{\bar{r}}\frac{\partial \bar{u}}{\partial \theta}=0 \qquad (5)$$

The boundary conditions of Eqs. (1), (3-5) are

$$\bar{u} = \bar{v} = 0\,,\ \bar{T} = \bar{T}_1 + k\bar{z}\,, \text{ on } \bar{r} = R \tag{6a}$$

$$\bar{u} = \bar{v} = 0\,,\ \bar{T} = \bar{T}_2 + k\bar{z}\,, \text{ on } \bar{r} = R + 2L \tag{6b}$$

where $T_m = (T_1 + T_2)/2$. We introduce the non-dimensional variables:

$$t = \frac{U_0}{\sigma L}\bar{t}\,,\ r = \frac{\bar{r}}{L}\,,\ z = \frac{\bar{z}}{L}\,,\ (u, v, w) = \frac{1}{U_0}(\bar{u}, \bar{v}, \bar{w})\,,\ T = \frac{\bar{T} - \bar{T}_0 - k\bar{z}}{\bar{T}_m}\,,\ U_0 = \frac{gK\beta T_0}{\nu} \tag{7}$$

where . The governing equations (1), (3-5) become

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

$$\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = \frac{\partial T}{\partial r}\,,\ \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \tag{9a-b}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z} - \frac{M^2}{Ra}w = \frac{1}{Ra}\left[\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right] \tag{10}$$

where $Ra = gK\beta T_0 L/\alpha_m \nu$ is the Rayleigh number and M is a parameter defined as: $M^2 = -kL^2 U_0/(\alpha_m \bar{T}_m)$. The boundary conditions (6) become

$$u = v = 0\,,\ T = -1, \text{ on } r = a \tag{11a}$$

$$u = v = 0\,,\ T = 1, \text{ on } r = a + 2 \tag{11b}$$

where $a = R/L$.

UNDISTURBED FIELD

If we take an axisymmetric field $v = v(r)$ as the basic field, Eqs. (8)-(10) then become

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0\,,\ \frac{\partial v}{\partial r} = 0\,,\ \frac{\partial w}{\partial r} = \frac{\partial T}{\partial r} \tag{12a-c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + M^2 w = 0 \tag{13}$$

subject to the boundary conditions

$$u = v = 0\,,\ T = -1, \text{ on } r = a \tag{14a}$$

$$u = v = 0\,,\ T = 1, \text{ on } r = a + 2 \tag{14b}$$

These equations give

$$u = v = 0\,,\ w = T + C \tag{15}$$

where C is a constant and Eq. (13) transforms to

$$\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + M^2 T = -M^2 C \tag{16}$$

If we take $x = Mr$, then Eq. (16) reduces to

$$\frac{d^2 T}{dx^2} + \frac{1}{x}\frac{dT}{dx} + T = -C \tag{17}$$

subject to

$$T = -1 \text{ at } x = Ma \tag{18a}$$

$$T = 1 \text{ at } x = M(a + 2) \tag{18b}$$

These boundary conditions are not sufficient to determine the solution of Eq. (17) uniquely because C still remains as a arbitrary constant. In order to obtain a unique solution, we impose an aditional condition of zero flux in the $\bar{z}$–direction. This condition is actually satisfied in a fluid layer of finite depth and we require the same condition in the limit of infinite depth. Then

$$\int_a^{a+2} r(T + C)\,dr = 0 \tag{19}$$

The general solution of Eq. (17) is

$$T = -C + AJ_0(Mr) + BY_0(Mr) \tag{20}$$

where A and B are undetermined constants yet. Using (18-19) and some properties of Bessel functions the values of the constants A, B and C are readily obtained.

STABILITY ANALYSIS

In order to examine the stability of the steady field

$$\mathbf{V}_s = (0,0,w(r)),\ T_s = T(r) \tag{21}$$

we superimpose the infinitesimal disturbances: $\hat{\mathbf{V}} = (\hat{u},\hat{v},\hat{w})$ and $\hat{T}$, to obtain

$$\mathbf{V} = \mathbf{V}_s + \hat{\mathbf{V}},\ T = T_s + \hat{T} \tag{22a}$$

We remark now that the undisturbed fields do not depend on θ and z, so that they can be decomposed into the Fourier components

$$(\hat{u},\hat{v},\hat{w},\hat{T}) = [\Phi(r),\psi(r),\chi(r),\tau(r)]\exp[i(\gamma z + n\theta - \omega t)] \tag{22b}$$

where γ denotes the wavenumber in the z-direction, n is an integer and $\omega = \omega_r + i\omega_i$ is a complex circular frequency, ω_r / γ being the phase velocity and ω_i is the amplification rate. According as ω_i is positive zero or negative, the disturbance is amplified, neutral or damped. Substituting (22a-b) into Eqs. (8) and (10) we have for $T_0 > 0$

$$\frac{1}{r}\frac{d}{dr}(r\Phi) + \frac{in}{r}\psi + i\gamma\chi = 0 \tag{23a}$$

$$\frac{d\chi}{dr} - i\gamma\Phi = \frac{d\tau}{dr} \tag{23b}$$

$$\frac{d\psi}{dr} + \frac{\psi}{r} - \frac{in}{r}\chi = 0 \tag{23c}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tau}{dr}\right) - \left[\gamma^2 + \frac{n^2}{r^2}W(r)\right]\tau + M^2\chi + \frac{i\Theta_s'}{\gamma}\Phi = 0 \tag{23d}$$

where $W(r) = i\gamma(w_s - c)$, $\Theta_s(r) = i\gamma RaT_s$ and $c = \omega/\gamma = c_r + ic_i$ is the wavespeed..

We solve now Eqs (23) subject to the boundary conditions

$$\Phi = \psi = \chi = \tau = 0 \text{ at } r = a \text{ and } r = a + 2 \tag{24}$$

Eliminating χ from Eqs. (23), we obtain

$$\chi = -i\frac{r}{n}\left(\frac{d\psi}{dr} + \frac{\psi}{r}\right) \tag{25}$$

and finally we have

$$\Phi'' + \frac{1}{r}\Phi + \frac{\gamma r}{n}\psi' + \left(\frac{in}{r} + \frac{\gamma}{n}\right)\tau = 0 \tag{26a}$$

$$\frac{r}{n}\psi'' + \frac{2}{n}\psi' - i\tau' + \gamma\Phi = 0 \tag{26b}$$

$$\tau'' + \frac{1}{r}\tau' - \frac{iM^2 r}{n}(\tau' + \tau) - \left[\gamma^2 + \frac{n^2}{r^2}W(r)\right]\tau + \frac{i\Theta_s'}{\gamma}\Phi = 0 \tag{26c}$$

subject to the boundary conditions

$$\Phi = \psi = \tau = 0 \text{ at } r = a \text{ and } r = a + 2 \tag{27}$$

The problem is now to solve the eigenvalue problem consisting of (26) and (27), where c is the eigenvalue and a, n, γ, Ra and M are parameters. The neutral curves (characterized by $c_i = 0$, i.e. a temporal analysis) are calculated for various values of the parameters a, n, γ, Ra and M. The strategy is to minimize the problem with respect to the Rayleigh number for fixed values of a, n,

γ and M. The numerical method used in this study for the stability analysis is the well-known compound matrix method.

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