HALF-SPACE ALBEDO PROBLEM IN A RAYLEIGH SCATTERING ATMOSPHERE WITH TRUE ABSORPTION

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ABSTRACT

The H_N method, employed for studies in neutron transport theory, is used to establish numerical results basic to the vector equation describing the transfer of polarized light in a Rayleigh scattering atmosphere with true absorption[1,2]. The method has been applied to the half-space albedo problem.

We consider the vector equation of transfer in plane geometry[3,4,]

$$\mu \frac{\partial \mathbf{I}(\tau,\mu)}{\partial \tau} + \mathbf{I}(\tau,\mu) = \frac{1}{2}\omega\,\mathbf{Q}(\mu)\int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu')\mathbf{I}(\tau,\mu')\,d\mu' \tag{1}$$

$\mathbf{I}(\tau,\mu)$ is a vector whose two components $\mathrm{I}_\ell(\tau,\mu)$ and $\mathrm{I}_r(\tau,\mu)$ are the angular intensities in the two states of polarization. τ is the optical variable, μ is the direction cosine (as measured from the positive τ axis) of the propagating radiation, and $\omega \in [0,1]$ is the single-scattering albedo. Here $\mathbf{Q}^{\mathrm{T}}(\mu)$ denotes the transpose of $\mathbf{Q}(\mu)$ [5]

$$\mathbf{Q}(\mu) = \frac{3}{2}(c+2)^{-1/2}\begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix}. \tag{2}$$

The parameter c, $c \in [0,1]$ is a measure of the Rayleigh component of scattering law: $c=1$ and $\omega=1$ would yield Chandrasekhar's conservative Rayleigh-scattering case, $\omega=1$ and $c\in[0,1]$ is Chandrasekhar's conservative case for a mixture of Rayleigh and isotropic scattering laws. $c=1$ and $\omega\in[0,1]$ yields general Rayleigh scattering[6].The general solution of Eq.(1) can be written as

$$\begin{aligned}\mathbf{I}(\tau,\mu) &= A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)e^{-\frac{\tau}{\eta_0}} + A(-\eta_0)\mathbf{\Phi}(-\eta_0,\mu)e^{\frac{\tau}{\eta_0}} \\ &\quad + \int_{-1}^{1}\left[A_1(\eta)\mathbf{\Phi}_1(\eta,\mu) + A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)\right]e^{-\frac{\tau}{\eta}}d\eta\end{aligned} \tag{3}$$

here $A(\pm\eta_0)$ and the vector $A_1(\eta), A_2(\eta)$ are the arbitrary expansion coefficients to be determined once the appropriate boundary conditions are specified[7]. The discrete and continuum normal modes are written respectively as

$$\mathbf{\Phi}(\pm\eta_0,\mu) = \frac{1}{2}\omega\eta_0\frac{1}{(\eta_0 \mp \mu)}\mathbf{Q}(\mu)\mathbf{M}(\pm\eta_0) \tag{4a}$$

$$\boldsymbol{\Phi}_{\alpha}(\eta,\mu)=\frac{1}{2}\omega\left[\eta\frac{\mathrm{P}}{\eta-\mu}+\lambda_{\alpha}(\eta)\delta(\eta-\mu)\right]\mathbf{Q}(\mu)\mathbf{M}_{\alpha}(\eta) \tag{4b}$$

Let us consider the half-space medium at $x>0$ and vacuum at $x<0$. In Eq.(1) neglecting $A(-\eta_0)$ and $A_1(-\eta), A_2(-\eta)$, we write the required solution as

$$\mathbf{I}(0,\mu)=A(\eta_0)\boldsymbol{\Phi}(\eta_0,\mu)+\int_0^1\left[A_1(\eta)\boldsymbol{\Phi}_1(\eta,\mu)+A_2(\eta)\boldsymbol{\Phi}_2(\eta,\mu)\right]d\eta \quad . \tag{5}$$

For the half-space albedo problem we seek the solution of Eq.(5) which satisfies the following boundary condition[8]

$$\mathbf{I}(0,\mu)=\delta(\mu-\mu_0)\mathbf{F} \quad , \quad \mu,\ \mu_0>0 \tag{6}$$

where $\mathbf{F}$ is a constant vector. The expansion of exit distribution can be chosen as[9]

$$\mathbf{I}(0,-\mu)=\mathbf{Q}(\mu)\sum_{\alpha=0}^{N}\mathbf{a}_{\alpha}\mu^{\alpha} \quad , \quad \mu>0 \tag{7}$$

To get the expansion coefficients in Eq.(5), it is multiplied by $\mu\boldsymbol{\Phi}^{\mathrm{T}}(\xi,\mu)$, $(\xi=\eta_0,\pm\eta)$, integrated over μ, $\mu\in(-1,+1)$ and used the orthogonality relations[5]

$$A(\eta_0)=-\frac{1}{N(\eta_0)}\left[\frac{\omega\eta_0}{2}\sum_{\alpha=0}^{N}\boldsymbol{\Gamma}_{\alpha}(\eta_0)\mathbf{a}_{\alpha}+\mu_0\boldsymbol{\Phi}^{T}(\eta_0,\mu_0)\mathbf{F}\right] \tag{8}$$

$$A_1(\eta)=\frac{1}{N(\eta)}\left[-\frac{\omega\eta}{2}\sum_{\alpha=0}^{N}\left(\mathrm{N}_{22}(\eta)\boldsymbol{\Gamma}_{\alpha}^{(1)}(\eta)-\mathrm{N}_{12}(\eta)\boldsymbol{\Gamma}_{\alpha}^{(2)}(\eta)\right)\mathbf{a}_{\alpha}+\mu_0\boldsymbol{\Phi}_1^{T^{\dagger}}(\eta,\mu_0)\mathbf{F}\right] \tag{9}$$

$$A_2(\eta)=\frac{1}{N(\eta)}\left[-\frac{\omega\eta}{2}\sum_{\alpha=0}^{N}\left(-\mathrm{N}_{21}(\eta)\boldsymbol{\Gamma}_{\alpha}^{(1)}(\eta)+\mathrm{N}_{11}(\eta)\boldsymbol{\Gamma}_{\alpha}^{(2)}(\eta)\right)\mathbf{a}_{\alpha}+\mu_0\boldsymbol{\Phi}_2^{T^{\dagger}}(\eta,\mu_0)\mathbf{F}\right] \tag{10}$$

here

$$\boldsymbol{\Gamma}_{\alpha}(\eta_0)=\frac{2}{\omega\eta_0}\int_0^1\mu^{\alpha+1}\boldsymbol{\Phi}^{\mathrm{T}}(\eta_0,-\mu)\mathbf{Q}(\mu)d\mu \tag{11a}$$

$$\boldsymbol{\Gamma}_{\alpha}^{\beta}(\eta)=\frac{2}{\omega\eta}\int_0^1\mu^{\alpha+1}\boldsymbol{\Phi}_{\beta}{}^{\mathrm{T}}(\eta,-\mu)\mathbf{Q}(\mu)d\mu, \qquad \beta=1,2\,. \tag{11b}$$

Now we consider the exit distribution in Eq.(5) replacing μ with $-\mu$ and multiplying it by μ^{m+1}, integrating over $\mu\in(0,1)$ we obtain

$$\begin{aligned}
&\sum_{\alpha=0}^{N}\Big\{\int_0^1\mu^{m+\alpha+1}\mathbf{Q}(\mu)d\mu+\left(\frac{\omega\eta_0}{2}\right)^2\frac{1}{N(\eta_0)}\mathbf{A}_m(\eta_0)\boldsymbol{\Gamma}_{\alpha}(\eta_0)\\
&-\left(\frac{\omega}{\eta}\right)^2\int_0^1\frac{\eta^2}{N(\eta)}\mathbf{A}_{\mathbf{m}}^{\mathbf{1}}(\eta)[-N_{22}(\eta)\boldsymbol{\Gamma}_{\alpha}^{1}(\eta)+N_{12}(\eta)\boldsymbol{\Gamma}_{\alpha}^{2}(\eta)]d\eta\\
&-\left(\frac{\omega}{\eta}\right)^2\int_0^1\frac{\eta^2}{N(\eta)}\mathbf{A}_{\mathbf{m}}^{\mathbf{2}}(\eta)[N_{21}(\eta)\boldsymbol{\Gamma}_{\alpha}^{1}(\eta)-N_{11}(\eta)\boldsymbol{\Gamma}_{\alpha}^{2}(\eta)]d\eta\Big\}\,\mathbf{a}_{\alpha} \qquad (12)\\
&=\frac{\omega\eta_0}{2}\mathbf{A}_m(\eta_0)\frac{1}{N(\eta_0)}\mu_0\boldsymbol{\Phi}^{T}(\eta_0,\mu_0)\mathbf{F}+\int_{\mathbf{0}}^{\mathbf{1}}\frac{\omega\eta}{2}\frac{1}{N(\eta)}\mu_0\left(\mathbf{A}_{\mathbf{m}}^{(\mathbf{1})}(\eta)\boldsymbol{\Phi}_1^{T^{\dagger}}(\eta,\mu)+\mathbf{A}_{\mathbf{m}}^{(\mathbf{2})}(\eta)\boldsymbol{\Phi}_2^{T^{\dagger}}(\eta,\mu)\right)\mathbf{F}\,d\eta
\end{aligned}$$

$$\mathbf{A}_{\alpha}(\eta_0) = \frac{2}{\omega\eta_0}\int_0^1 \mu^{\alpha+1}\mathbf{\Phi}(-\eta_0,\mu)d\mu$$
$$\mathbf{A}_{\alpha}^{\beta}(\eta) = \frac{2}{\omega\eta}\int_0^1 \mu^{\alpha+1}\mathbf{\Phi}_{\alpha}^{\beta}(-\eta_0,\mu)d\mu \quad , \quad \beta = 1,2 \tag{13}$$

The numerical results of the required albedo values, as functions of the single scattering albedo and c which is a measure of the Rayleigh component of scattering law, can be obtained from the following equation

$$\beta = \int_0^1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \mathbf{I}(0,-\mu)\,\mu\,d\mu \Bigg/ \int_0^1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \mathbf{I}(0,\mu)\,\mu\,d\mu \tag{14}$$

where $\mathbf{a}_{\boldsymbol{\alpha}}$ can be found from Eq. (12).

REFERENCES

1. Tezcan, C. Kaşkaş, A. and Güleçyüz, M. Ç., The H_N Method for Solving Linear Transport Equation: Theory and Applications, *J. Quant. Spectrosc. Radiat. Transfer*, Vol. 78, pp 243-254, 2003.
2. Karahasanoğlu Şenyiğit, M. and Kaşkaş, A., The H_N Method for Milne Problem with Polarization, *Astrophysics and Space Science,* (In press) .
3. Mourad, S., A. and Siewert, C., E., On Solutions of An Equation of Transfer for A Planetary Atmosphere, *Astrophys. J*, Vol. 155, pp 555-564, 1969.
4. Schnatz, T., W. and Siewert, C., E., Radiative Transfer in A Rayleigh-Scattering Atmosphere with True Absorption, *J. Math.Phys,* Vol. 11, pp 2733-2739, 1970.
5. Bond, G. R. and Siewert, C. E., On the Nonconservative Equation of Transfer for A Combination of Rayleigh and Isotropic Scattering, *Astrophys. J*, Vol. 164, pp 97-110, 1971.
6. Siewert, C. E., A Concise and Accurate Solution for A Polarization Model in Radiative Transfer, *J. Quant. Spectrosc. Radiat. Transfer*, Vol. 62, pp 677-684, 1999.
7. Burniston, E. E. and Siewert, C., E., Half-Range Expansion Theorems in Studies of Polarized Light, *J. Quant. Spectrosc. Radiat. Transfer*, Vol. 11, pp 3416-3420, 1970.
8. Schnatz, T. W. and Siewert, C., E., On the Transfer of Polarized Light in Rayleigh-Scattering Half Spaces with True Absorption, *Mon. Not. R. Astr. Soc.,* Vol. 152, pp 491-508, 1971.
9. Siewert, C. E., On Using the F_N Method for Polarization Studies in Finite Plane-parallel Atmospheres, *J. Quant. Spectrosc. Radiat. Transfer,* Vol. 21, pp 35-39, (1979).